Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 1, pp. 113–119, 1996
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0021–8928/96 $24.00+0.00

S0021–8928(96)00015-9

A REFINED THEORY OF THE VIBRATIONS OF A CYLINDRICAL SHELL BASED ON AN EXPANSION IN SERIES OF THE NORMAL DISPLACEMENT†

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(*Received* 11 *January* 1994)

A refined theory of vibrations of a multilayer orthotropic cylindrical shell based on the method of hypotheses [1, 2] and an expansion of the normal displacement in series in terms of the thickness of the shell [3] is considered. The fact that the results are intended to be used in practical structures determines the set of selected hypotheses. © 1996 Elsevier Science Ltd. All rights reserved.

Applied problems require a theory enabling one to describe the wave process from a single standpoint both in the two spatial directions parallel to the surface of the shell and in one direction around a region cut out from the shell.

The purpose of this paper is to construct a theory of the vibrations of a cylindrical shell and a zone of finite dimensions in which we take into account the transverse deformation and the possibility of wave processes being generated by this effect.

Existing results cannot be used directly to solve this problem because of the contradictions inherent in the methods of solution. One of the contradictions is that the well-known method of [3] stems from representing the normal displacement as a series in terms of the thickness of a flat shell, and the resulting solution describes vibrations in one spatial direction. In [3] there is no indication of how the results can be used to describe the propagation of wave processes in two spatial directions in curved shells. On the other hand, the method used in [1] employs only the null term of the expansion of the normal displacement of the shell and describes the propagation of the wave process in two spatial directions. In this case the resulting general equations of motion contain a contradiction, which can be readily seen when the equations are applied to a flat shell. The equations of motion of a flat shell [2, p. 238] admit of vibrations generated by a symmetric external load when there are no transverse deformation and antisymmetric vibrations, which physically should not be the case.

In the present paper, by removing these contradictions, we can obtain an extended system of Ambartsumyan equations, which describe the wave processes in a shell of finite dimensions.

Consider a rigid cylindrical shell consisting of an odd number $(2m + 1)$ of orthotropic layers arranged symmetrically about the middle one, which will be numbered as zero ($i = 0$). The layers above the middle one are given positive numbers from $i = 1$ to $i = m$, while those below the middle one are numbered from $i = -1$ to $i = -m$. Layers arranged symmetrically about the middle one have the same thickness and elasticity parameters.

The directions of an orthogonal system of coordinates α, β, γ coincide with the principal anisotropic axes of the elastic material. The origin is in the middle of the length, width and thickness of the shell. The coordinate α is taken to be the central angle of a transverse arc drawn from some initial rectilinear generatrix, β being the length of the generatrix. Once the dimensions of α and β are chosen in this way, we have

$$H_1 = R_c(1 + k\gamma), \quad H_2 = 1, \quad k = 1/R_c$$

for the Lamé coefficients. Here R_c is the radius of curvature of the middle cylindrical surface.

The following hypotheses are used:

1. the shear stresses vary across the thickness of the shell as prescribed by [2, p. 46]

$$\tau_{\alpha\gamma i} = G_H \varphi f(\gamma), \quad \varphi \equiv \varphi(\alpha, \beta, t)$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 1, pp. 120–126, 1996.

$$\tau_{\beta\gamma i} = G_H \chi f(\gamma), \quad \chi \equiv \chi(\alpha, \beta, t)$$

$$f(\gamma) = f(-\gamma), \quad f(\gamma)\big|_{\gamma=\pm h} = 0$$

2. the normal displacement can be expanded in series in terms of the thickness of the shell [3]

$$U_{\gamma i} = \frac{1}{h}(R_c a\alpha + c\beta) + \sum_{j=0}^{2N+1} W_j \left(\frac{\gamma}{h}\right)^j, \quad W_j = W_j(\alpha, \beta, t)$$

3. the tangential displacements of the middle surface are given by [2, p. 33]

$$U_{\alpha 0}(\alpha, \beta, \gamma, t)\big|_{\gamma=0} = b + U, \quad U \equiv U(\alpha, \beta, t)$$

$$U_{\beta 0}(\alpha, \beta, \gamma, t)\big|_{\gamma=0} = d + V, \quad V \equiv V(\alpha, \beta, t)$$

4. the shell is loaded on the outer and inner cylindrical surfaces, the end surfaces being free of any loads.

Here α, β, γ are orthogonal coordinates, t is the time, $2h$ is the thickness of the many-layer shell, a and c are unknown angles of rotation of the shell as a whole, b and d are unknown displacements of the shell as a whole, $\tau_{\alpha\gamma i}$, $\tau_{\beta\gamma i}$ are the shear stresses in the ith layer corresponding to the appropriate coordinate axes, $U_{\alpha i}$, $U_{\beta i}$, $U_{\gamma i}$, are the displacements, $f(\gamma)$ is a given function characterizing the variation of the shear stresses across the thickness of the shell, φ and χ are unknown functions, W_j is an unknown component of the normal displacement in the expansion in terms of the thickness of the shell, U and V are the tangential displacements of the middle surface caused by the vibrations of the shell, $2N + 1$ is the number of terms in the expansion, and G_H is a convenient normalization coefficient.

The equilibrium conditions for the medium of the ith layer undergoing harmonic vibrations can be represented by the following differential equations in a cylindrical system of coordinates [1, p. 18]

$$\begin{aligned}
&\sigma_{\alpha i,\alpha} + (H_1\tau_{\alpha\beta i})_{,\beta} + (H_1\tau_{\alpha\gamma i})_{,\gamma} + \tau_{\alpha\gamma i} + \rho_i\omega^2 H_1 U_{\alpha i} = 0 \\
&\sigma_{\beta i,\beta} + \tau_{\beta\gamma i,\gamma} + \frac{1}{H_1}\tau_{\beta\gamma i} + \frac{1}{H_1}\tau_{\alpha\beta i,\alpha} + \rho_i\omega^2 U_{\beta i} = 0 \\
&(H_1\sigma_{\gamma i})_{,\gamma} + \tau_{\alpha\gamma i,\alpha} + (H_1\tau_{\beta\gamma i})_{,\beta} + \rho_i\omega^2 H_1 U_{\gamma i} = 0
\end{aligned} \tag{1}$$

Here i is the number of the layer (i varies from $-m$ to m), p_i is the density of the material of the layer, ω is the angular velocity, $\sigma_{\alpha i}$, $\sigma_{\beta i}$, $\sigma_{\gamma i}$,are the principal stresses, and $\tau_{\alpha\beta i}$ is the shear stress. The harmonic factor $e^{-i\omega t}$ is omitted in all formulae.

When deriving the equations of motion the simplicity of the equations and the precision of comparative results between shells is taken to be important in applied problems, rather than the precise description of wave processes in a shell. Therefore, when using the first equation of equilibrium of the medium in (1), we shall neglect the terms of order $(kh)^2$ compared to unity [1, p. 122]. When using the second equation in (1), we shall neglect the terms of order (kh) compared to unity in all expressions. An approximate condition is used as the third condition of equilibrium of the medium in (1), in which we neglect the weak influence of $\sigma_{\alpha i}$ on the stress $\sigma_{\gamma i}$ normal to the shell, which is to be determined.

In (1) the propagation of the wave process along the central transverse arc is described more accurately than along the generatrix. Nevertheless, the use of a single approach to describe shells having the same inner structure but different surface wave parameters enables us to obtain correct comparative results in applied problems.

Substituting into the third condition in (1) the given laws governing the variation of the shear stresses and the displacement normal to the median surface, integrating the result with respect to γ from γ_i to γ (γ being inside the ith layer), and taking into account the equality of normal stresses on the boundaries of the layer, we can find an expression from which to determine the normal component of the principal stress at any point of the multilayer shell

$$(1 + k\gamma)\,\sigma_{\gamma i} = \sigma - G_H\left[JX_0(\gamma)\frac{1}{R_c}\frac{\partial\varphi}{\partial\alpha} + JZ_0(\gamma)\frac{\partial\chi}{\partial\beta}\right] -$$

$$-\omega^2 h\left[\frac{R_c a\alpha + c\beta}{h}(RZ_{i0}+\rho_i[I_{i0}(\gamma)+khI_{i1}(\gamma)])+\right.$$

$$\left.+\sum_{j=0}^{2N+1} W_j(RZ_{ij}+\rho_i[I_{ij}(\gamma)+khI_{i,j+1}(\gamma)])\right] \tag{2}$$

$$I_{ij}(\gamma)=\frac{1}{h}\int_{\gamma_i}^{\gamma}\left(\frac{\gamma}{h}\right)^j d\gamma,\quad JZ_i(\gamma)=\int_{\gamma_i}^{\gamma}(1+k\gamma)\,f(\gamma)\,d\gamma$$

$$JZ_i(\gamma)=JX_i(\gamma)+khJY_i(\gamma),\quad RZ_{ij}\equiv RZ_{i,j}$$

$$RZ_{\pm s,j}=RZ_{\pm s\mp 1,j}+[I_{\pm s\mp 1,j}(\gamma_{\pm s})+khI_{\pm s\mp 1,j+1}(\gamma_{\pm s})]\,\rho_{\pm s\mp 1}$$

$$RZ_{\pm s,j}=RX_{\pm s,j}+khRY_{\pm s,j},\quad RZ_{0j}=0,\quad \gamma_0=0,\quad s=1,2,\dots,m$$

Here σ is an unknown integration constant, $\gamma_1, \gamma_2, \dots, \gamma_{m+1}$ are the coordinates of the upper boundaries of the layers with numbers $0, 1, \dots, m$, and $\gamma_{-1}, \gamma_{-2}, \dots, \gamma_{-m+1}$ are the coordinates of the lower boundaries of the layers with numbers $0, -1, \dots, -m$. The number RZ_{ij} can be computed from the recurrent formula involving upper and lower mathematical symbols. The upper symbols correspond to positive values of i, while the lower ones correspond to negative values.

Taking into account relations (2), the fact that $f(\gamma)$ is an even function, and the symmetry of the layers about the middle one, we can find an equation of equilibrium of the shell apart from terms $O(kh)$. Differentiating the equilibrium equation with respect to α, we find the first equation of motion of the shell

$$\frac{1}{2}\frac{\partial}{\partial\alpha}(\sigma_H-\sigma_b)-\frac{kh}{2}\frac{\partial}{\partial\alpha}(\sigma_H+\sigma_b)=\frac{2}{3}G_H\left(\frac{1}{R_c}\frac{\partial^2\varphi}{\partial\alpha^2}+\frac{\partial^2\chi}{\partial\alpha\partial\beta}\right)+$$

$$+\omega^2 R_c[aRX_{m+1,0}+(\psi_0 RX_{m+1,0}+\ldots+\psi_{2N}RX_{m+1,2N})+$$

$$+kh(\psi_1 RY_{m+1,1}+\ldots+\psi_{2N+1}RY_{m+1,2N+1})],\quad \psi_j=\frac{h}{R_c}\frac{\partial W_j}{\partial\alpha} \tag{3}$$

Differentiating the equilibrium equation with respect to β and neglecting terms $O(kh)$, we find another equation of motion of the shell

$$\frac{1}{2}\frac{\partial}{\partial\beta}(\sigma_H-\sigma_b)=\frac{2}{3}G_H\left(\frac{1}{R_c}\frac{\partial^2\varphi}{\partial\alpha\partial\beta}+\frac{\partial^2\chi}{\partial\beta^2}\right)+$$

$$+\omega^2[cRX_{m+1,0}+(\xi_0 RX_{m+1,0}+\ldots+\xi_{2N}RX_{m+1,2N})]$$

$$\xi_j=h\frac{\partial W_j}{\partial\beta} \tag{4}$$

Here σ_H and σ_b are the normal stresses on the lower and upper boundaries of the multilayer shell and ψ_j and ξ_j are the unknown components of the angles of rotation about the corresponding coordinate axes in the expansion in terms of the thickness of the shell.

The stress state of the orthotropic medium of the ith layer can be determined by the semi-inverse method of the theory of elasticity from Hooke's law [1, p. 16] without using the equation determining the normal deformation

$$\sigma_{\alpha i}=B_{11}^i e_{\alpha i}+B_{12}^i e_{\beta i}+A_1^i\sigma_{\gamma i} \tag{5}$$

$$\sigma_{\beta i}=B_{21}^i e_{\alpha i}+B_{22}^i e_{\beta i}+A_2^i\sigma_{\gamma i}$$

$$\tau_{\alpha\gamma i} = G_{1i} e_{\alpha\gamma i}, \quad \tau_{\beta\gamma i} = G_{2i} e_{\beta\gamma i}, \quad \tau_{\alpha\beta i} = G_{3i} e_{\alpha\beta i}$$
$$B_{11}^i = E_1^i / \Delta, \quad B_{22}^i = E_2^i / \Delta, \quad B_{12}^i = \nu_{12}^i E_1^i / \Delta, \quad B_{12}^i = B_{21}^i$$
$$A_1^i = \frac{E_1^i}{E_3^i} \frac{\nu_{13}^i + \nu_{12}^i \nu_{23}^i}{\Delta}, \quad A_2^i = \frac{E_2^i}{E_3^i} \frac{\nu_{23}^i + \nu_{13}^i \nu_{21}^i}{\Delta}$$
$$\Delta = 1 - \nu_{12}^i \nu_{21}^i$$

Here $e_{\alpha i}$, $e_{\beta i}$, are the volume deformations in the ith layer along the corresponding coordinate axes, $e_{\alpha\gamma i}$, $e_{\beta\gamma i}$, $e_{\alpha\beta i}$, are the shear deformations, ν_{12}^i, ν_{13}^i, ν_{21}^i, ν_{23}^i are Poisson's ratios E_1^i, E_2^i, E_3^i are Young's moduli, and G_{1i}, G_{2i}, G_{3i}, are the shear moduli.

From the first two equations (5) we can see that A'_1 and A'_2 are small compared with B_{11}^i, B_{12}^i, B_{22}^i and the contribution of the normal stress component two $\sigma_{\alpha i}$ and $\sigma_{\beta i}$ is small. Therefore, when determining $\sigma_{\gamma i}$ and the integration constant from (2) one can neglect infinitesimal higher-order terms

$$\sigma_{\gamma i} = \sigma - G_H J X_0(\gamma)\left(\frac{1}{R_c}\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\chi}{\partial\beta}\right) -$$
$$-\omega^2 h\left[\frac{R_c a\alpha + c\beta}{h})(RX_{i0} + I_{i0}(\gamma)\rho_i) + \sum_{j=0}^{2N+1} W_j (RX_{ij} + I_{ij}(\gamma)\rho_i)\right] \tag{6}$$
$$\sigma = \sigma_c + \omega^2 h(W_1 RX_{m+1,1} + \ldots + W_{2N+1} RX_{m+1,2N+1}), \quad \sigma_c = \frac{1}{2}(\sigma_H + \sigma_b)$$

Solving the fourth and fifth equations of (5), we obtain expressions for the shear displacements at any point of the multilayer shell, neglecting terms $O((kh)^2)$

$$U_{\alpha i} = (1 + k\gamma)(b + U) - a[(1 + k\gamma) I_{00}(\gamma) - 2kh I_{01}(\gamma)] -$$
$$-\sum_{j=0}^{2N+1} [(1 + k\gamma) I_{0j}(\gamma) - 2kh I_{0,j+1}(\gamma)]\psi_j + \varphi\left[(1 + k\gamma)\left(AX_i + JX_i(\gamma)\frac{1}{g_{1i}}\right) +\right.$$
$$\left. + kh\left(AY_i - JY_i(\gamma)\frac{1}{g_{1i}}\right)\right]$$

$$U_{\beta i} = d + V - cI_{00}(\gamma) - \sum_{j=0}^{2N+1} \xi_j I_{0j}(\gamma) + \chi\left(BZ_i + JX_i(\gamma)\frac{1}{g_{2i}}\right) \tag{7}$$
$$AZ_0 = 0, \quad AZ_{\pm s} = AX_{\pm s} + khAY_{\pm s}, \quad s = 1, 2, \ldots, m$$
$$AZ_{\pm s} = AZ_{\pm s\mp 1} + [JX_{\pm s\mp 1}(\gamma_{\pm s}) - khJY_{\pm s\mp 1}(\gamma_{\pm s})]\frac{1}{g_{1i}}$$
$$BZ_0 = 0, \quad BZ_{\pm s} = BZ_{\pm s\mp 1} + JX_{\pm s\mp 1}(\gamma_{\pm s})\frac{1}{g_{2i}}$$
$$g_{ni} = \frac{G_{ni}}{G_H}, \quad n = 1, 2$$

The relative deformations of the ith layer of the shell along the corresponding coordinate axes are given by [1, p. 18]

$$e_{\alpha i} = \frac{1}{H_1}(U_{\alpha i,\alpha} + U_{\gamma i}), \quad e_{\beta i} = U_{\beta i,\beta} \tag{8}$$

$$e_{\alpha i}=\frac{1}{R_c}\frac{\partial U}{\partial\alpha}-\sum_{j=0}^{2N+1}(I_{0j}(\gamma)-2khI_{0,j+1}(\gamma))\frac{1}{R_c}\frac{\partial\psi_j}{\partial\alpha}+$$
$$+\frac{1}{R_c}\frac{\partial\varphi}{\partial\alpha}-\left[AX_i+JX_i(\gamma)\frac{1}{g_{1i}}+kh\left(AY_i-JY_i(\gamma)\frac{1}{g_{1i}}\right)\right]+$$
$$+\frac{(1-k\gamma)}{R_c}\left(\frac{R_c a\alpha+c\beta}{h}+\sum_{j=0}^{2N+1}W_j\left(\frac{\gamma}{h}\right)^j\right)$$
$$e_{\beta i}=\frac{\partial V}{\partial\beta}-\sum_{j=0}^{2N+1}I_{0j}(\gamma)\frac{\partial\xi_j}{\partial\beta}+\frac{\partial\chi}{\partial\beta}\left(BZ_i+JX_i(\gamma)\frac{1}{g_{2i}}\right)$$

The shear deformation of the ith layer of the shell [1, p. 18] is equal to

$$e_{\alpha\beta i}=U_{\alpha i,\beta}+\frac{1}{H_1}U_{\beta i,\alpha}\tag{9}$$
$$e_{\alpha\beta i}=(1+k\gamma)\frac{\partial U}{\partial\beta}-\sum_{j=0}^{2N+1}[(1+k\gamma)I_{0j}(\gamma)-2khI_{0,j+1}(\gamma)]\frac{\partial\psi_j}{\partial\beta}+$$
$$+\frac{\partial\varphi}{\partial\beta}\left[(1+k\gamma)\left(AX_i+JX_i(\gamma)\frac{1}{g_{1i}}\right)+kh\left(AY_i-JY_i(\gamma)\frac{1}{g_{1i}}\right)\right]+$$
$$+\frac{(1-k\gamma)}{R_c}\left[\frac{\partial V}{\partial\alpha}-\sum_{j=0}^{2N+1}I_{0j}(\gamma)\frac{\partial\xi_j}{\partial\alpha}+\frac{\partial\chi}{\partial\alpha}\left(BZ_i+JX_i(\gamma)\frac{1}{g_{2i}}\right)\right]$$

When substituting $e_{\alpha i}$ and $e_{\beta i}$ into the second equation of equilibrium of the medium (1), we neglect terms $O(kh)$.

Using the first two equations of equilibrium of the medium (1), Eqs (3)–(9), and the equivalence conditions for the moments acting in the cross-sections of the shell, we obtain the equations of motion of the shell

$$\frac{1}{R_c^2}\frac{\partial^2}{\partial\alpha^2}D\begin{Vmatrix}Z_1\\Z_2\end{Vmatrix}+\frac{\partial^2}{\partial\beta^2}Q\begin{Vmatrix}Z_1\\Z_2\end{Vmatrix}+\omega^2R\begin{Vmatrix}Z_1\\Z_2\end{Vmatrix}=\varphi HX+MX\begin{Vmatrix}a\\b\end{Vmatrix}+$$
$$+\frac{\partial^2}{R_c\partial\alpha\partial\beta}\left(MY\begin{Vmatrix}Y_1\\Y_2\end{Vmatrix}+MZ\begin{Vmatrix}Z_1\\Z_2\end{Vmatrix}\right)+\begin{Vmatrix}TX\dfrac{1}{R_c}\dfrac{\partial}{\partial\alpha}(\sigma_H+\sigma_b)\\0\\\vdots\\0\\\dfrac{1}{2R_c}\dfrac{\partial}{\partial\alpha}(\sigma_H-\sigma_b)-\dfrac{kh}{2R_c}\dfrac{\partial}{\partial\alpha}(\sigma_H+\sigma_b)\end{Vmatrix}$$
$$\frac{\partial^2}{R_c^2\partial\alpha^2}C\begin{Vmatrix}Y_1\\Y_2\end{Vmatrix}+\frac{\partial^2}{\partial\beta^2}P\begin{Vmatrix}Y_1\\Y_2\end{Vmatrix}+\omega^2G\begin{Vmatrix}Y_1\\Y_2\end{Vmatrix}=\chi HY+NX\begin{Vmatrix}c\\d\end{Vmatrix}+$$

$$
+\frac{\partial^2}{R_c\partial\alpha\partial\beta}\left(NY\begin{Vmatrix} Y_1 \\ Y_2 \end{Vmatrix}+NZ\begin{Vmatrix} Z_1 \\ Z_2 \end{Vmatrix}\right)+\begin{Vmatrix} TY\dfrac{\partial}{\partial\beta}(\sigma_H+\sigma_b) \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{2}\dfrac{\partial}{\partial\beta}(\sigma_H-\sigma_b) \end{Vmatrix} \tag{10}
$$

$$
D=\begin{Vmatrix} DX & kh\,DY \\ kh\,DZ & DT \end{Vmatrix},\quad Q=\begin{Vmatrix} QX & kh\,QY \\ kh\,QZ & QT \end{Vmatrix},\quad R=\begin{Vmatrix} RX & kh\,RY \\ kh\,RZ & RT \end{Vmatrix}
$$

$$
C=\begin{Vmatrix} CX & 0 \\ 0 & CT \end{Vmatrix},\quad P=\begin{Vmatrix} PX & 0 \\ 0 & PT \end{Vmatrix},\quad G=\begin{Vmatrix} GX & 0 \\ 0 & GT \end{Vmatrix}
$$

$$
D=(D_{js}),\quad Q=(Q_{js}),\quad R=(R_{js}),\quad HX=(HX_s),\quad TX=(TX_n)
$$

$$
MX=(MX_{is}),\quad MY=(MY_{js}),\quad MZ=(MZ_{js})
$$

$$
C=(C_{js}),\quad P=(P_{js}),\quad G=(G_{js}),\quad HY=(HY_s),\quad TY=(TY_n)
$$

$$
NX=(NX_{is}),\quad NY=(NY_{js}),\quad NZ=(NZ_{js})
$$

$$
Z_1'=(U,\psi_1,\ldots,\psi_{2N+1}),\quad Z_2'=(\varphi,\psi_0,\ldots,\psi_{2N})
$$

$$
Y_1'=(V,\xi_1,\ldots,\xi_{2N+1}),\quad Y_2'=(\chi,\xi_0,\ldots,\xi_{2N})
$$

$$
j,s=1,2,\ldots,2N+4,\quad i=1,2,\quad n=N+2
$$

Here Z_1, Z_2, Y_1, Y_2 are column-matrix functions and the prime denotes transposition. For brevity, we will not present the explicit form of the matrix coefficients.

The solution of the system of equations (10) is sought in terms of two pairs of unknown vectors (Z_1, Z_2) and (Y_1, Y_2). Each pair of vectors describes the vibrations of the shell along the corresponding coordinate axis and in the plane normal to the shell surface. Symmetric vibrations of the shell along the corresponding coordinate axes are described by Z_1 and Y_1, while antisymmetric vibrations are described by Z_2 and Y_2. The dimension of each linear system of differential equations is determined by the terms of the expansion of the normal displacement in terms of the shell thickness. Matrix terms being linear functions of (kh) are collected in the first system of equations (10), but they are discarded in the second system of equations.

If we restrict ourselves only to the null term in the expansion of the normal displacement in terms of the thickness, then system (10) does not reduce to the known equations [1, 2]. The difference is that the equations of motion are written in terms of the unknown angles of rotation of the shell, rather than in terms of the unknown displacements. The new form of the equations of the shell proves important when wave processes are considered. It enables one to obtain the correct order of the differential wave equations and the number of boundary conditions when the number of unknowns is increased.

From the system of equations (10) one can easily obtain the equations of motion of an elongated cylindrical region lying along a transverse arc of the cylindrical surface, the width of which is several times greater than the thickness. As a special case, from (10) one can obtain the equations of motion of a region along the generatrix of the cylindrical surface. The wave process in the region manifests itself strongly only down the length of the region. In this case one of the systems of equations in (10) becomes identically equal to zero. When deriving the equations of motion of the region one can prescribe the laws governing the variation of the shear stresses as a function of the thickness and width [2, p. 46], if necessary. The normal displacement given as a series in terms of the thickness [3] is assumed to be independent of the region width, i.e. there is no wave process across the region width. Since the dependence on the region width occurs in the equations through the given law governing the variation of the shear stress, the resulting equations must be averaged by integrating them over the width of the region. In this case Eqs (10) will be expressed in terms of averaged values.

The general solution of problem (10) for a shell can be sought as a series in terms of the column-matrix eigenfunctions [4, p. 78] which satisfy homogeneous matrix equations describing the free vibrations of the shell and homogeneous boundary conditions

$$
\begin{gathered}
(L+\omega_n^2 R)\begin{Vmatrix} Z_{1n} \\ Z_{2n} \end{Vmatrix} = \begin{Vmatrix} 0 \\ \vdots \\ 0 \end{Vmatrix}, \quad L = \frac{1}{R_c^2}\frac{\partial^2}{\partial\alpha^2}D + \frac{\partial^2}{\partial\beta^2}Q \\
(M_s + \varepsilon_{sl}^2 GX)\,Y_{1l} = \begin{Vmatrix} 0 \\ \vdots \\ 0 \end{Vmatrix}, \quad (M_a + \varepsilon_{am}^2 GT)\,Y_{2m} = \begin{Vmatrix} 0 \\ \vdots \\ 0 \end{Vmatrix} \\
M_s = \frac{1}{R_c^2}\frac{\partial^2}{\partial\alpha^2}CX + \frac{\partial^2}{\partial\beta^2}PX, \quad M_a = \frac{1}{R_c^2}\frac{\partial^2}{\partial\alpha^2}CT + \frac{\partial^2}{\partial\beta^2}PT
\end{gathered}
\tag{11}
$$

Here n, l, m are the numbers of the characteristic modes of vibration, ω_n, ε_{sl}, ε_{am} are the characteristic angular frequencies, Z_{1n}, Z_{2n}, Y_{1l}, Y_{2m} are the column-matrix eigenfunctions, L is a matrix differential operator, and M_s and M_a are the matrix differential operators of symmetric and antisymmetric vibrations.

The theory of vibrations of a cylindrical shell constructed above, which is based on an expansion of the normal displacement in series in terms of the thickness, can be used in applied problems, in which the transverse compression of the shell must be taken into account. The results enable shells having the same inner structure and different surface wave dimensions to be compared.

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Translated by T.J.Z.